6.2

This Is Series(ous) Business

Finite Arithmetic Series

LEARNING GOALS

In this lesson, you will:

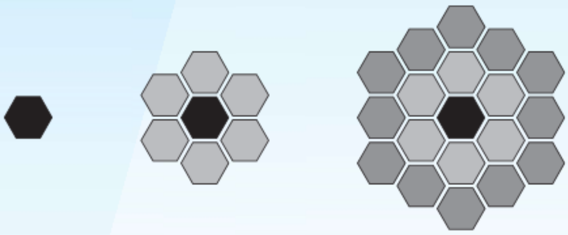
- Compute a finite series.
- Use sigma notation to represent a sum of a finite series.
- Use Gauss's method to compute finite arithmetic series.
- Write a function to represent the sum of a finite arithmetic series.
- Use finite arithmetic series to solve real-world problems.

KEY TERMS

- tessellation
- series
- finite series
- infinite series
- arithmetic series

Honey bees are fascinating little creatures. Did you know that honey bees are the only insects that produce food that humans eat? They also identify members of their colony by a unique smell.

Another amazing aspect of the honey bee is how they build their hive. Honey bees build hexagonal honey cells from a single cell. Layers of honey cells are then built around the edges, as shown.



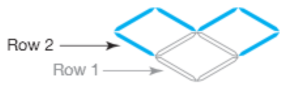
Could you sketch the next figure in the sequence? Could you predict how many total hexagons would be in the next term of the sequence? How does this pattern translate to an arithmetic sequence?

PROBLEM 1 Project: Toothpick Tessellation

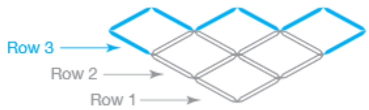
Josephine is helping her little brother Pauley with his latest art project. He is using toothpicks to create a *tessellation*. A **tessellation** is created when a geometric shape is repeated over a two-dimensional plane such that there are no overlaps and no gaps.

Pauley starts his tessellation project by gluing toothpicks to a large piece of poster board to make a single diamond shape. This is the first row.

Then, he places additional toothpicks parallel to the first row to create the second row. The second row consists of two diamond shapes.



He continues to place toothpicks in this manner, so that each row has one more diamond shape than the previous row. The first three rows of Pauley's tessellation are shown.



1. Sketch the next two rows of the tessellation on the previous diagram.

2. Complete the table to show the number of additional toothpicks used to create each row.

Row	Number of Additional Toothpicks Used to Create the Row
1	
2	
3	
4	
5	

3. Does this tessellation represent an arithmetic or geometric sequence? Explain how you know.

4. Write an explicit formula for this sequence. Let n represent the row number, and let a_n represent the number of additional toothpicks used to create that row.

5. Suppose that Pauley knows that he wants his tessellation to include 18 rows. How many additional toothpicks will he need for the 18th row? Explain how you determined your answer.

6. Describe how to calculate the total number of toothpicks that Pauley needs for a tessellation that includes 18 rows. (Do not actually perform the calculation.)

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You know how to determine the *n*th term of a sequence. However, sometimes it is necessary to determine the *sum* of the terms in a sequence.

A **series** is the sum of terms in a given sequence. The sum of the first n terms of a sequence is denoted by S_n. For example, S_3 is the sum of the first three terms of a sequence.

There is a special notation for the summation of terms using a capital sigma, Σ:

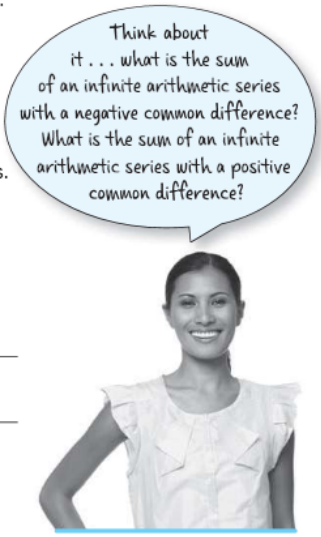
$$S_n = \sum_{i=1}^{n} a_i$$

- upper bound of summation: n
- an indexed variable representing each successive term in the series: a_i
- index of summation: i
- lower bound of summation: 1

This expression means sum the values of a, starting at a_1 and ending with a_n.

In other words, $S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$.

A series can be *finite* or *infinite*. A **finite series** is the sum of a finite number of terms. An **infinite series** is the sum of an infinite number of terms. For example, the sum of all of the even integers from 1 to 100 is a finite series, and the sum of all of the even whole numbers is an infinite series.



7. Use sigma notation to rewrite each finite series, and then compute.

a. 5 + 9 + 13 + 17 + 21

S_2 = ______________________

S_5 = ______________________

b. 3 + 6 + 12 + 24 + 48 + 96 + 192

S_1 = ______________________

S_7 = ______________________

8. Use sigma notation to represent the total number of toothpicks Pauley needs to complete 5 rows of his tessellation. Then, use your table in Question 2 to calculate this amount.

PROBLEM 2 Gauss's Method to the Rescue!

Remember that an arithmetic sequence is a sequence of numbers in which the difference between any two consecutive terms is a constant. An **arithmetic series** is the sum of an arithmetic sequence.

You can compute a finite arithmetic series by adding each individual term, but this can take a lot of time. A famous mathematician named Carl Friedrich Gauss developed another way to compute a finite arithmetic series.

As the story goes, when Gauss was in elementary school, his teacher asked the class to calculate the sum of the first 100 positive integers. Apparently, Gauss determined the answer in a matter of seconds! How did Gauss determine his answer so quickly?

1. Complete the steps and answer the questions to see how Gauss was able to calculate the sum of the first 100 positive integers so quickly.

   a. The series S_{100} is shown. The same series in descending order is shown beneath it. Add the series by computing the sum of each pair of vertical, or partial sums.

$$S_{100} = 1 + 2 + 3 + \cdots + 98 + 99 + 100$$
$$+S_{100} = 100 + 99 + 98 + \cdots + 3 + 2 + 1$$
$$2S_{100} = \_\_\_\_ + \_\_\_\_ + \_\_\_\_ + \cdots + \_\_\_\_ + \_\_\_\_ + \_\_\_\_$$

   b. What do you notice about each partial sum?

   c. How many partial sums are there in this series?

   d. Write the sum of the partial sums.

   $2S_{100}$ = ______________________

   e. To arrive at the total in part (d), you actually added each term of the series twice. How could you calculate the correct total from the sum of the partial sums, or S_{100}?

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   f. What is S_{100}?

   S_{100} = ______________________

Gauss's method can be generalized for any finite arithmetic series.

2. Consider a finite arithmetic series S_n written as a sum of its terms.

$$S_n = a_1 + a_2 + a_3 + \cdots + a_{n-2} + a_{n-1} + a_n$$

Complete the steps shown to determine Gauss's formula to compute any finite arithmetic series.

a. First, write S_n in terms of a_1, a_n, and the common difference d. Remember that for an arithmetic sequence, $a_n = a_1 + d(n - 1)$.

$$S_n = a_1 + (\_\_\_\_\_\_) + (a_1 + 2d) + \cdots + (a_n - 2d) + (\_\_\_\_\_\_) + \_\_\_\_\_\_$$

b. Then, write S_n in reverse order.

$$S_n = \_\_\_\_\_\_ + (a_n - d) + (\_\_\_\_\_\_) + \cdots (a_1 + 2d) + (\_\_\_\_\_\_) + a_1$$

c. Add the series, keeping the "+" and "=" signs vertically aligned.

$$S_n = a_1 + (\_\_\_\_) + (\_\_\_\_) + \cdots + (\_\_\_\_) + (\_\_\_\_) + a_n$$

$$+S_n = a_n + (\_\_\_\_) + (\_\_\_\_) + \cdots + (\_\_\_\_) + (\_\_\_\_) + a_1$$

$$2S_n = (\_\_\_\_) + (\_\_\_\_) + (\_\_\_\_) + \cdots + (\_\_\_\_) + (\_\_\_\_) + (\_\_\_\_)$$

d. Identify each partial sum.

e. Fill in the blanks to show the sum of the partial sums.

$$2S_n = \_\_\_\_\_\_(a_1 + \_\_\_\_\_\_)$$

f. Fill in the blanks to write the formula for S_n.

$$S_n = \frac{\_\_\_\_\_\_(a_1 + \_\_\_\_\_\_)}{2}$$

g. Describe Gauss' rule to compute any finite arithmetic series by completing the sentence.

Add the ________________ term and the ________________ term of the series, multiply the sum by the number of ________________ of the series, and divide by ________________.

So, Gauss's formula to compute the first n terms of an arithmetic series is shown.

$$S_n = \frac{n(a_1 + a_n)}{2}$$

3. Use the *Toothpick Tessellation* problem situation to answer each question.

   a. Use Gauss's formula to calculate the total number of toothpicks Pauley needs to complete 5 rows of his tessellation. Show your work.

   b. Remember that Pauley wanted his tessellation to include a total of 18 rows. If he has a box of 350 toothpicks, does he have enough? Explain why or why not.

PROBLEM 3 Human Calculator, or Inspiration from Gauss?

In the previous problem, you learned a way to compute the first n-terms of any finite arithmetic series. Now, you will take a closer look at some special series of numbers.

1. Consider a sequence of odd whole numbers.

   a. Write an explicit formula to determine any term of the sequence.

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b. Use Gauss's formula to calculate the sum of the first 20 odd whole numbers. What information did you need to use Gauss's formula?

Emma claims that she can calculate the sum of the first 20 odd whole numbers using a different method. Emma's method does not require her to calculate the last term of the series.

2. Let's determine how Emma can perform this calculation.

a. Substitute the known value of a_1 and the algebraic expression for a_n into Gauss's formula.

b. Write your answer from part (a) using function notation. Describe the function.

c. Use your function from part (b) to calculate the sum of the first 20 odd whole numbers. Verify that your result the same as your result in Question 1.

3. Calculate the sum of the first 100 odd whole numbers. Then, verify your result using Gauss's formula.

What about even numbers? You can use a similar process to compute the series of even whole numbers.

4. Follow the given steps to calculate the sum of the first 100 even whole numbers.

   a. Write an explicit formula to calculate any term of the sequence of even whole numbers.

   b. Substitute the known value of a_1 and the algebraic expression for a_n into Gauss's formula.

   c. Write your answer from part (b) using function notation.

   d. Use your function from part (c) to calculate the first 100 even whole numbers. Then, verify using Gauss's formula.

5. Compare the function for the series of even whole numbers with the function for the series of odd whole numbers. What makes them different? Explain why you think the difference exists.

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PROBLEM 4 "Chair"-ity Case

You are in charge of setting up for your high school band's annual Spring concert. The concert will be held outdoors on the school soccer field, and one of your duties is to arrange the seating for the show.

You have gathered the following information.

- The stage is 20 feet wide.
- The first row of chairs will be about the same width as the stage.
- Each successive row of chairs will have three more chairs than the previous row. This way, the chairs are offset so that each person does not have a chair directly in front of them for better viewing.
- Each chair is 1.5 feet wide and 1.5 feet deep.
- There needs to be 0.5 foot of spacing in between the chairs within a row so that the audience can sit comfortably.
- In order to have enough room for people to walk through the rows, there needs to be 4 feet of space in between each row, from the back of one chair to the back of the other chair.

Use the given information to answer each question.

1. What factors do you need to consider when determining how many rows of chairs there could be?

2. How many chairs are in the first row? Explain your reasoning.

3. Sketch a seating chart that includes the given information and dimensions.

Answer each question based on the additional given information. Show all your work.

4. Suppose that the first 5 rows of chairs make up the “gold circle” section.

a. How many chairs are in the gold circle section?

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b. How many feet deep is the gold circle section?

5. Suppose that you need a total of 500 chairs for the concert.

   a. How many rows will you need with this number of chairs?

   b. How deep is the seating area with this number of chairs?

6. Suppose that no row can have more than 40 chairs.

   a. What is the maximum number of rows possible?

   b. What is the maximum number of people that can be seated?

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Be prepared to share your solutions and methods.